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Dual Baxter equations and quantization of the affine Jacobian

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Abstract. A quantum integrable model is considered which describes a quantization of the affine hyper-elliptic Jacobian. This model is shown to possess the property of duality: a dual model with inverse Planck constant exists such that the eigenfunctions of its Hamiltonians coincide with the eigenfunctions of Hamiltonians of the original model. We explain that this duality can be considered as duality between homologies and cohomologies of quantized affine hyper-elliptic Jacobian.

1. Introduction

The relation between the affine Jacobian and integrable models is well known (cf [1]). In paper [2] we have shown that the algebra of functions on the affine Jacobian is generated by the action of Hamiltonian vector fields from a finite number of functions. The latter functions are coefficients of the highest non-vanishing cohomologies of the affine Jacobian. Actually, the idea that such a description of the algebra of functions is possible appeared in [3] which considers the structure of the algebra of observables for the quantum and the classical Toda chain.

In this paper we give a quantum version of [2]. A quantum mechanical model is formulated which gives a quantization of the affine Jacobian. As usual in quantum mechanics, we can describe not the variety itself but the algebra of functions on it (observables). We need to show that the quantum algebra of observables possesses the essential property of the corresponding classical algebra of functions. In our case this property is the possibility of creating every observable from a finite number of observables (cohomologies) by the action of Hamiltonians.

In the process of realization of this programme we find the Baxter equations which describe the spectrum of the model. It happens that these equations possess the property of duality: there is a dual model with inverse Planck constant for which the eigenvectors are the same. The algebras of observables of two dual models commute. The next ingredient of our study is the method of separation of variables developed by Sklyanin [4]. Using this method we present the matrix elements of any observable in terms of certain integrals.

We show that the integrals in question are expressed in terms of deformed Abelian integrals (cf [3, 5]). The observables for both dual models are defined in terms of cohomologies. The most beautiful feature of our construction is that in these cohomologies the integrals for matrix elements enter in such a way that the cohomologies of the dual model play the role

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of homologies for the original one and vice versa. We consider this relation between weak–strong duality in quantum theory and duality between homologies and cohomologies as the most important conclusion of this paper.

2. Affine Jacobian

In this section we briefly summarize necessary facts concerning the relation between integrable models and algebraic geometry, following paper [2]. The reason for repeating certain facts from [2] is that we shall need them in a slightly different situation.

Consider a 2×2 matrix which depends polynomially on the parameter z:

$$m(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

where the matrix elements are polynomials of the form

$$\begin{aligned} a(z) &= z^{g+1} + a_1 z^g + \cdots + a_{g+1} \\ b(z) &= z^g + b_1 z^{g-1} + \cdots + b_g \\ c(z) &= c_2 z^g + c_3 z^{g-1} \cdots + c_{g+2} \\ d(z) &= d_2 z^{g-1} + d_3 z^{g-2} \cdots + d_{g+1}. \end{aligned} \tag{1}$$

In the the affine space $\mathbb{C}^{4g+2}$ with coordinates $a_1, \ldots, a_{g+1}$, $b_1, \ldots, b_g$, $c_2, \ldots, c_{g+2}$, $d_2, \ldots, d_{g+1}$ consider the $(2g+1)$-dimensional affine variety $\mathcal{M}$ defined as quadric

$$f(z) \equiv a(z)d(z) - b(z)c(z) = 1. \tag{2}$$

We consider this simplest situation, but in principle it is possible to put an arbitrary polynomial of degree $2g$ in the rhs.

On the quadric $\mathcal{M}$ let us consider the sections $J_{\mathrm{aff}}(t)$ defined by

$$a(z) + d(z) = t(z) \tag{3}$$

where $t(z)$ is given polynomial of the form

$$t(z) = z^{g+1} + z^g t_1 + \cdots + t_{g+1}. \tag{4}$$

The notation $J_{\mathrm{aff}}(t)$ stands for affine Jacobi variety. The definition of affine Jacobi variety and its equivalence to $J_{\mathrm{aff}}(t)$ described above are given in the appendix A. We include appendix A because there is minor difference with the situation considered in [1, 2]. The variety $\mathcal{M}$ is foliated into the affine Jacobians $J_{\mathrm{aff}}(t)$. The mechanical model described below provides a clever way of describing this foliation.

We wish to understand the geometrical meaning of quantum integrable models. The general philosophy teaches that in order to describe the quantization of a manifold one has to deform the algebra of functions on this manifold, preserving certain essential properties of this algebra. The classical algebra must allow the Poisson structure in order that quantization is possible.

Certain Poisson brackets for the coefficients of matrix $m(z)$ can be introduced. We do not write them down explicitly; if needed they can be obtained by taking the classical limit of the commutation relations (14). The algebra

$$\hat{\mathcal{A}} = \mathbb{C}[a_1, \ldots, a_{g+1}, b_1, \ldots, b_g, c_2, \ldots, c_{g+2}, d_2, \ldots, d_{g+1}]$$

becomes a Poisson algebra. The most important properties of this Poisson structure are the following. First, the coefficients of the determinant $f(z)$ belong to the centre of the Poisson algebra, so equation (2) is consistent with Poisson structure. Second, the trace $t(z)$ generates the commutative subalgebra

$$\{t(z), t(z')\} = 0.$$

In fact it can be shown that the coefficient t_{g+1} of the trace belongs to the centre; it is convenient to put $t_{g+1} = (-1)^{g+1}2$. The subject of our study is the algebra of functions on the $\mathcal{M}$:

$$\mathcal{A} = \frac{\hat{\mathcal{A}}}{\{f(z) = 1, t_{g+1} = (-1)^{g+1}2\}}$$

on which the Poisson structure is well defined.

The Poisson commutative algebra generated by the coefficients $t_1, \ldots, t_g$ is called the algebra of the integrals of motion. Introduce the commuting vector fields

$$\partial_i g = \{t_i, g\} \qquad i = 1, \ldots g.$$

The vector fields ∂_j describe motion along the subvarieties $J_{\mathrm{aff}}(t)$.

One can think of these vector fields as

$$\partial_j = \frac{\partial}{\partial \tau_j}$$

where τ_j are 'times' corresponding to the integrals of motion t_j. Define the ring of integrals of motion

$$\mathcal{T} = \mathbb{C}[t_1, \ldots, t_g]. \tag{5}$$

Introduce the space of differential forms C^k with basis

$$x\, \mathrm{d}\tau_{i_1} \wedge \cdots \wedge \mathrm{d}\tau_{i_k} \qquad x \in \mathcal{A}$$

and the differential

$$d = \partial_j\, \mathrm{d}\tau_j.$$

Consider the corresponding cohomologies H^k. In [2] the arguments are given in favour of the following conjecture.

Conjecture 1. *The cohomologies H^k are finite-dimensional over the ring $\mathcal{T}$; they are isomorphic to the cohomologies of the affine variety $J_{\mathrm{aff}}(t)$ with t in generic position.*

On the algebra $\mathcal{A}$ and on the spaces C^k one can introduce degree [2]. Take the basis of H^g considered as a vector space over $\mathcal{T}$ which is composed of homogeneous representatives

$$\Omega_\alpha = g_\alpha\, \mathrm{d}\tau_1 \wedge \cdots \wedge \mathrm{d}\tau_g$$

where α takes a finite number of values. The fact of foliation of $\mathcal{M}$ into varieties $J_{\mathrm{aff}}(t)$ corresponds to the following statement concerning the algebra $\mathcal{A}$ [2].

Proposition 1. *Every element x of $\mathcal{A}$ can be presented in the form*

$$x = \sum_\alpha p_\alpha(\partial_1, \ldots, \partial_g) g_\alpha \tag{6}$$

where $p_\alpha(\partial_1, \ldots, \partial_g)$ are polynomials of $\partial_1, \ldots, \partial_g$ with coefficients in $\mathcal{T}$.

The representation (6) is not unique; the equations

$$\sum_\alpha p_\alpha(\partial_1, \ldots, \partial_g) g_\alpha = 0 \tag{7}$$

are counted by H^{g-1} [2].

Formula (6) can be useful only if we are able to control the cohomologies. Concerning these cohomologies we adopt several conjectures following [2].

3. Conjectured form of cohomologies

The affine variety $J_{\mathrm{aff}}(t)$ allows the following description. Consider the hyper-elliptic curve X of genus g:

$$w^2 - t(z)w + 1 = 0. \tag{8}$$

This curve has two points over the point $z = \infty$ which we denote by $\infty^{\pm}$.

Consider a matrix $m(z)$ satisfying (2). Take the zeros of $b(z)$:

$$b(z) = \prod_{j=1}^{g}(z - z_j)$$

and

$$w_j = d(z_j).$$

Obviously z_j, w_j satisfy the equation of the curve X (8). Thus $m(z)$ defines a point $\mathcal{P}$ (divisor) on the symmetrized gth power $X[g]$ of the curve X. The divisor $\mathcal{P}$ consists of the points $p_j = (z_j, w_j) \in X$.

On the other hand, one can reconstruct $m(z)$ starting form the divisor $\mathcal{P}$. The corresponding map is singular, the singularities being located on

$$D = \{\mathcal{P} | p_i = \sigma(p_j) \text{ for some } i, j \text{ or } p_i = \infty^{\pm} \text{ for some } i\} \tag{9}$$

where σ is a hyper-elliptic involution. Thus the alternative description of $J_{\mathrm{aff}}(t)$ is

$$J_{\mathrm{aff}}(t) = X[g] - D.$$

Consider the meromorphic differentials on X with singularities at $\infty^{\pm}$. We choose the following basis of these differentials:

$$\begin{aligned} \mu_k(p) &= z^{g+k}\frac{\mathrm{d}z}{y} \qquad && -g \leqslant k \leqslant 0 \\ \mu_k(p) &= \left[y\frac{\mathrm{d}}{\mathrm{d}z}(z^{k-g-1}y)\right]_{\geqslant}\frac{\mathrm{d}z}{y} \qquad && k \geqslant 1 \end{aligned} \tag{10}$$

where $p = (z, w)$, $y = 2w - t(z)$, $[\;]_{\geqslant}$ means that only non-negative degrees of Laurent series in the brackets are taken.

The form

$$\tilde{\mu}_k = \sum_i \mu_k(p_i)$$

is viewed as a form on $J_{\mathrm{aff}}(t)$. It is easy to see that the forms μ_k (hence $\tilde{\mu}_k$) with $k \geqslant g + 1$ are exact. Consider the space W^m with the basis

$$\Omega_{k_1,\ldots,k_m} = \tilde{\mu}_{k_1} \wedge \cdots \wedge \tilde{\mu}_{k_m}$$

where $-g \leqslant k_j \leqslant g$. As in [2] we adopt the following conjecture.

Conjecture 2. *We have*

$$H^m = \frac{W^m}{\sigma \wedge W^{m-2}} \tag{11}$$

where

$$\sigma = \sum_{j=1}^{g}\tilde{\mu}_j \wedge \tilde{\mu}_{-j}.$$

According to (9) the singularities of differential forms occur either at $p_i = \sigma(p_j)$ or at $p_i = \infty^\pm$. The nontrivial essence of conjecture 2 is that the first kind of singularity can be eliminated by adding exact forms. There are $(g-1)$-forms singular at $p_i = \sigma(p_j)$ such that these singularities disappear after applying d. This is the origin of the space $\sigma \wedge W^{k-2}$ [2].

Consider briefly the dual picture. On the affine curve with punctures at $\infty^\pm$ there are $2g+1$ nontrivial cycles δ_k with $k = -g, \ldots, g$. The cycles δ_k, $k < 0$ are a-cycles, the cycles δ_k, $k > 0$ are b-cycles and δ_0 is the cycle around ∞^+. One defines the cycles $\tilde{\delta}_k$ on the symmetrical power of the affine curve. The $\wedge$-operation is introduced for these cycles by duality with cohomologies. The nontrivial consequence of conjecture 2 is that every cycle on $J_{\mathrm{aff}}(t)$ can be constructed by wedging $\tilde{\delta}_k$. The formula dual to (11) is

$$H_m = \frac{W_m}{\sigma' \wedge W_{m-2}} \tag{12}$$

where W_m is spanned by

$$\Delta_{k_1,\ldots,k_m} = \tilde{\delta}_{k_1} \wedge \cdots \wedge \tilde{\delta}_{k_m}$$

and

$$\sigma' = \sum_{j=1}^{g} \tilde{\delta}_j \wedge \tilde{\delta}_{-j}.$$

We need to factorize over $\sigma' \wedge W_{m-2}$ because the 2-cycle σ' intersects with D.

Let us return to the relation of H^g to the algebra $\mathcal{A}$. Notice that

$$\mathrm{d}\tau_1 \wedge \cdots \wedge \mathrm{d}\tau_g \simeq \tilde{\mu}_1 \wedge \cdots \wedge \tilde{\mu}_g \equiv \Omega.$$

The functions

$$x_{k_1,\ldots,k_g} = \Omega^{-1}\Omega_{k_1,\ldots,k_g}$$

are symmetric polynomials of $z_1, \ldots, z_g$. Recall that $b_1, \ldots, b_g$ are nothing but elementary symmetric polynomials of $z_1, \ldots, z_g$. Hence the coefficients of cohomologies have the form

$$g_\alpha = g_\alpha(b_1, \ldots, b_g).$$

The dimension of H^g is determined by conjecture 2:

$$\alpha = 1, \ldots, \binom{2g+1}{g} - \binom{2g+1}{g-2}.$$

Equations (7) are the consequences of the following ones:

$$\sum_{k=1}^{g} \partial_k(\Omega^{-1}(\mu_{-k} \wedge \Omega_{k_1,\ldots,k_{g-1}})) = 0 \qquad \forall \Omega_{k_1,\ldots,k_{g-1}} \in W^{g-1}. \tag{13}$$

4. Quantization of the affine Jacobian

Let us consider a quantization of algebra $\mathcal{A}$. The parameter of deformation (Planck constant) is denoted by γ, we shall also use

$$q = \mathrm{e}^{\mathrm{i}\gamma}.$$

Consider the 2×2 matrix $\boldsymbol{m}(z)$ with noncommuting entries. Suppose that the dependence on the spectral parameter z is exactly the same as in the classical case (1). The variables $\boldsymbol{a}_j$, $\boldsymbol{b}_j$, $\boldsymbol{c}_j$, $\boldsymbol{d}_j$ are subject to commutation relations which are summarized as follows:

$$r_{21}(z_1, z_2)\boldsymbol{m}_1(z_1)k_{12}(z_1)s_{12}\boldsymbol{m}_2(z_2)k_{21}(z_2) = \boldsymbol{m}_2(z_2)k_{21}(z_2)s_{21}\boldsymbol{m}_1(z_1)k_{12}(z_1)r_{12}(z_1, z_2) \tag{14}$$

where usual conventions are used: equation (14) is written in the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$, $a_1 = a \otimes I$, $a_2 = I \otimes 2$, $a_{21} = Pa_{12}P$ where P is the operation of permutations. The $\mathbb{C}$-number matrices r, k, s are

$$
\begin{aligned}
&r_{12}(z_1, z_2) = \frac{z_1 - qz_2}{1-q}(I \otimes I) + \frac{z_1 + qz_2}{1+q}(\sigma^3 \otimes \sigma^3) + 2(z_1\sigma^- \otimes \sigma^+ + z_2\sigma^+ \otimes \sigma^-)\\
&k_{12}(z) = I \otimes (I - \sigma^3) + (q^{-\sigma^3} + z(q^2-1)\sigma^-) \otimes (I + \sigma^3)\\
&s_{12} = I \otimes I - (q - q^{-1})\sigma^- \otimes \sigma^+.
\end{aligned}
\tag{15}
$$

These commutation relations are important because they respect the form of matrix $\boldsymbol{m}(z)$ prescribed by (1); we shall explain how they are related to more usual r-matrix relations in the next section.

Define the polynomials:

$$
\begin{aligned}
&\boldsymbol{t}(z) = q\boldsymbol{a}(z) + q^2\boldsymbol{d}(z) - z(q^2-1)\boldsymbol{b}(z)\\
&\boldsymbol{f}(z) = q\boldsymbol{d}(z)\boldsymbol{t}(zq^{-2}) - q^2\boldsymbol{d}(z)\boldsymbol{d}(zq^{-2}) - q\boldsymbol{b}(z)\boldsymbol{c}(zq^{-2}).
\end{aligned}
\tag{16}
$$

The algebra $\hat{\mathcal{A}}(q)$ is generated by $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{g+1}$, $\boldsymbol{b}_1, \dots, \boldsymbol{b}_g$, $\boldsymbol{c}_2, \dots, \boldsymbol{c}_{g+2}$, $\boldsymbol{d}_2, \dots, \boldsymbol{d}_{g+1}$. The polynomial $\boldsymbol{f}(z)$ belongs to the centre of $\hat{\mathcal{A}}(q)$. The coefficients of $\boldsymbol{t}(z)$ are commuting; in fact $\boldsymbol{t}_{g+1}$ belongs to the centre of $\hat{\mathcal{A}}(q)$. We define

$$
\mathcal{A}(q) = \frac{\hat{\mathcal{A}}(q)}{\{\boldsymbol{f}(z) = 1, \boldsymbol{t}_{g+1} = (-1)^{g+1}2\}}.
$$

The noncommutative algebra $\mathcal{A}(q)$ defines a quantization of the algebra of functions on the quadric $\mathcal{M}$. However, we cannot directly define the quantization of the algebra of functions on the affine Jacobian because the coefficients of $\boldsymbol{t}(z)$ are not in the centre of $\mathcal{A}(q)$. What we can do is to describe the quantum version of proposition 1 and of the description of cohomologies. The exposition will be more detailed than in the classical case.

As in [3] we accept the following conjecture.

Conjecture 3. *The algebra $\mathcal{A}(q)$ is spanned as linear space by elements of the form:*

$$
x = p_L(\boldsymbol{t}_1, \dots, \boldsymbol{t}_g)g(\boldsymbol{b}_1, \dots, \boldsymbol{b}_g)p_R(\boldsymbol{t}_1, \dots, \boldsymbol{t}_g) \tag{17}
$$

where $p_L(\boldsymbol{t}_1, \dots, \boldsymbol{t}_g)$, $g(\boldsymbol{b}_1, \dots, \boldsymbol{b}_g)$, $p_R(\boldsymbol{t}_1, \dots, \boldsymbol{t}_g)$ are polynomials.

We were not able to prove this statement; however, since the algebra $\mathcal{A}(q)$ is graded we can check it degree by degree. This has been done up to degree 8. Notice the similarity between the representation (17) and the representation for spin operators proved in [6]. Conjecture 3 implies that certain generalization of the results of [6] is possible. In fact, formula (17) is similar to formula (6): we can either symmetrize or antisymmetrize $\boldsymbol{t}_j$ in (17), which corresponds in classics to multiplication by t_j or to applying ∂_j. In order to have complete agreement with the classical case we have to show that only finitely many different polynomials $g(\boldsymbol{b}_1, \dots, \boldsymbol{b}_g)$ (cohomologies) create the entire algebra $\mathcal{A}(q)$.

Notice that the commutation relations (14) imply in particular that

$$
[\boldsymbol{b}(z), \boldsymbol{b}(z')] = 0
$$

which means that we have the commutative family of operators $\boldsymbol{z}_j$ defined by

$$
\boldsymbol{b}(z) = \prod(z - \boldsymbol{z}_j).
$$

So, every polynomial $g(\boldsymbol{b}_1, \dots, \boldsymbol{b}_g)$ can be considered as a symmetric polynomial of $\boldsymbol{z}_j$ and vice versa.

It is very convenient to use the following formal definitions. Consider the ring $\mathcal{T}$ defined in (5). By $\mathcal{V}^k$ we denote the space of antisymmetric polynomials of k variables such that their degrees with respect to every variable is not less than 1 with coefficients in $\mathcal{T} \otimes \mathcal{T}$. In other words, $\mathcal{V}^k$ is the space spanned by the polynomials

$$p_L \cdot h \cdot p_R \equiv p_L(t_1, \ldots, t_g) h(z_1, \ldots, z_k) p_R(t'_1, \ldots, t'_g)$$

where h is antisymmetric, vanishing when one of z_j vanishes. The following operations can be defined.

(1) Multiplication by t_j and t'_j.

(2) Operation $\wedge : \mathcal{V}^k \otimes \mathcal{V}^l \to \mathcal{V}^{k+l}$ which is defined as follows:

$$(p_L \cdot h \cdot p_R) \wedge (p'_L \cdot h' \cdot p'_R) = p_L p'_L \cdot (h \wedge h') \cdot p_R p'_R$$

where

$$(h \wedge h')(z_1, \ldots, z_{k+l}) = \frac{1}{k!l!} \sum_{\pi \in S_{k+l}} (-1)^\pi h(z_{\pi(1)}, \ldots, z_{\pi(k)}) h'(z_{\pi(k+1)}, \ldots, z_{\pi(k+l)}).$$

We have a map

$$\mathcal{V}^g \xrightarrow{\chi} \mathcal{A}(q)$$

defined on the basis elements as

$$\chi(p_L \cdot h \cdot p_R) = p_L(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_g) \frac{h(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_g)}{\prod \boldsymbol{z}_i \prod_{i<j} (\boldsymbol{z}_i - \boldsymbol{z}_j)} p_R(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_g)$$

and continued linearly. The conjecture 3 states that this map is surjective. We want to describe the kernel of the map χ.

First, consider the space $\mathcal{V}^1$. The elements of this space are polynomials of one variable z with coefficients in $\mathcal{T} \otimes \mathcal{T}$. In appendix B we describe a certain basis in $\mathcal{V}^1$ considered as a linear space over $\mathcal{T} \otimes \mathcal{T}$. The basis in question consists of the polynomials: s_k with $k \geqslant -g$ such that the degree of s_k with respect to z equals $g + k + 1$. The kernel of χ is the joint of three subspaces; let us describe them.

(1) For $k \geqslant g + 1$ we have

$$\chi(s_k \wedge \mathcal{V}^{g-1}) = 0. \tag{18}$$

(2) Consider $c \in \mathcal{V}^2$ defined as

$$c = \sum_{j=1}^{g} s_j \wedge s_{-j}.$$

We have

$$\chi(c \wedge \mathcal{V}^{g-2}) = 0. \tag{19}$$

(3) Consider $d \in \mathcal{V}^1$ defined as

$$d = (t_j - t'_j) s_{-j}.$$

We have

$$\chi(d \wedge \mathcal{V}^{k-1}) = 0. \tag{20}$$

The construction of the space

$$\frac{\mathcal{V}^g}{\mathrm{Ker}\ (\chi)} \simeq \mathcal{A}(q)$$

is in complete correspondence with the classical case. In classics we start with all the 1-forms $\tilde{\mu}_k$. Imposing (18) corresponds to throwing away the exact forms and working with $\tilde{\mu}_k$ for $k = -g, \ldots, g$ only. Imposing (19) corresponds to factorizing over $\sigma \wedge W^{k-2}$ in classics. Finally, (20) corresponds to equation (13).

The origin of equations (18)–(20) will be explained in section 9. There should be a purely algebraic method of proving these equations, but we do not know it. It is important to mention that by accepting conjecture 3 we are forced to conclude that the kernel of χ is completely described by the equations (18)–(20). This is proved by calculation of characters in a similar way as in [3].

5. The realization of $\mathcal{A}(q)$

We want to describe a realization of the algebra $\mathcal{A}(q)$ in a space of functions. Consider the quantum mechanical system described by the operators x_j with $j = 1, \ldots, 2g+2$ and y (zero mode). The operators x_j and y are self-adjoint, they satisfy the commutation relations:

$$x_k x_l = q^2 x_l x_k \qquad k < l$$
$$y x_k = q^2 x_k y \qquad \forall k.$$

The Hamiltonian of the system is

$$\boldsymbol{h} = q^{-1} \sum_{k=1}^{2g+2} x_k x_{k-1}^{-1}$$

where

$$x_{2g+3} \equiv q y x_1.$$

Physically this model defines the simplest lattice regularization of the chiral Bose field with modified energy–momentum tensor.

It is useful to double the number of degrees of freedom. Consider the algebra A generated by two operators u and v satisfying the commutation relations

$$uv = qvu.$$

Take the algebra $A^{\otimes(2g+2)}$; the operators u_j, v_j $(j = 1, \ldots, 2g+2)$ are defined as u and v acting in jth tensor component. The original operators x_i are expressed in terms of u_i, v_i as follows:

$$x_k = v_k \prod_{j=1}^{k-1} u_j^{-2} \qquad y = \prod_{j=1}^{2g+2} u_j.$$

Consider the 'monodromy matrix'

$$\tilde{\boldsymbol{m}}(z) = \begin{pmatrix} \tilde{\boldsymbol{a}}(z) & \tilde{\boldsymbol{b}}(z) \\ \tilde{\boldsymbol{c}}(z) & \tilde{\boldsymbol{d}}(z) \end{pmatrix} = l_{2g+2}(z) \ldots l_1(z) \tag{21}$$

where the l-operators are

$$l(z) = \frac{1}{\sqrt{z}} \begin{pmatrix} zu & -qvu \\ zv^{-1}u^{-1} & 0 \end{pmatrix}. \tag{22}$$

This is a particular case of the more general l-operator $l(z, \kappa)$ in which the last matrix element is not 0 but $\kappa z u$; the model corresponding to the latter l-operator is a subject of study in a series of papers [7].

The matrix elements of the matrix $\boldsymbol{m}(z)$ satisfy the commutation relations

$$r_{12}(z_1, z_2)\tilde{\boldsymbol{m}}_1(z_1)\tilde{\boldsymbol{m}}_2(z_2) = \tilde{\boldsymbol{m}}_2(z_2)\tilde{\boldsymbol{m}}_1(z_1)r_{12}(z_1, z_2) \tag{23}$$

where the r-matrix $r_{12}(z_1, z_2)$ is defined earlier (15). These are canonical r-matrix commutation relations. The quantum determinant of the matrix $\tilde{\boldsymbol{m}}(z)$ is defined by

$$\boldsymbol{f}(z) = \tilde{\boldsymbol{d}}(z)\tilde{\boldsymbol{a}}(zq^{-2}) - \tilde{\boldsymbol{b}}(z)\tilde{\boldsymbol{c}}(zq^{-2})$$

and it belongs to the centre; in our realization of $\tilde{\boldsymbol{m}}(z)$ one has $\boldsymbol{f}(z) = 1$. The trace of $\tilde{\boldsymbol{m}}(z)$ generates commuting quantities; we denote this trace as follows:

$$\tilde{\boldsymbol{a}}(z) + \tilde{\boldsymbol{d}}(z) = y\boldsymbol{t}(z).$$

The matrix elements of the matrix $\tilde{m}(z)$ are of the form

$$\begin{aligned}
\tilde{\boldsymbol{a}}(z) &= \tilde{\boldsymbol{a}}_0 z^{g+1} + \tilde{\boldsymbol{a}}_1 z^g + \cdots + \tilde{\boldsymbol{a}}_{g+1} \\
\tilde{\boldsymbol{b}}(z) &= \tilde{\boldsymbol{b}}_0 z^g + \tilde{\boldsymbol{b}}_1 z^{g-1} + \cdots + \tilde{\boldsymbol{b}}_g \\
\tilde{\boldsymbol{c}}(z) &= \tilde{\boldsymbol{c}}_1 z^{g+1} + \tilde{\boldsymbol{c}}_2 z^g + \cdots + \tilde{\boldsymbol{c}}_{g+1} z \\
\tilde{\boldsymbol{d}}(z) &= \tilde{\boldsymbol{d}}_1 z^g + \tilde{\boldsymbol{d}}_2 z^{g-1} + \cdots + \tilde{\boldsymbol{d}}_g z
\end{aligned} \tag{24}$$

where, in particular, $\tilde{\boldsymbol{a}}_0 = y$. This form of polynomial, $\tilde{\boldsymbol{a}}(z)$, $\tilde{\boldsymbol{b}}(z)$, $\tilde{\boldsymbol{c}}(z)$, $\tilde{\boldsymbol{d}}(z)$, does not correspond to what we have in the classical model of the affine Jacobian. This is the reason for modifying the matrix $\tilde{m}(z)$ as follows:

$$\boldsymbol{m}(z) = \begin{pmatrix} \tilde{\boldsymbol{a}}_0\tilde{\boldsymbol{b}}_0^{-1} & 0 \\ -\tilde{\boldsymbol{d}}_1\tilde{\boldsymbol{b}}_0^{-1} & 1 \end{pmatrix} \tilde{\boldsymbol{m}}(z) \begin{pmatrix} \tilde{\boldsymbol{b}}_0\tilde{\boldsymbol{a}}_0^{-1} & 0 \\ q\tilde{\boldsymbol{d}}_1\tilde{\boldsymbol{a}}_0^{-1} & 1 \end{pmatrix}.$$

The matrix elements of this matrix have structure (1), they satisfy closed commutation relations (14), and the operators $\boldsymbol{f}(z)$ and $\boldsymbol{t}(z)$ defined for these two matrices coincide; in particular we have

$$\boldsymbol{t}_1 = \boldsymbol{h}.$$

Thus the modification of matrix $\tilde{\boldsymbol{m}}(z)$ which is necessary for relation to the affine Jacobian is responsible for the appearance of the strange-looking commutation relations (14).

6. Q-operator

Our first goal is to define Baxter's Q-operator. Let us realize the operators v, u in $L_2(\mathbb{R})$ as follows:

$$v = \mathrm{e}^{\varphi} \qquad u = \mathrm{e}^{\mathrm{i}\gamma\frac{\mathrm{d}}{\mathrm{d}\varphi}}.$$

We shall work in the φ-representation, i.e. in the space $\mathfrak{H} = (L_2(\mathbb{R}))^{\otimes(2g+2)}$. Following the standard procedure [8] one introduces the vectors $Q(\zeta \mid \psi_1, \ldots, \psi_{2g+2})$ which depend on

$$\zeta = \tfrac{1}{2}\log z$$

and $2g+2$ additional parameters, ψ_j, and satisfy the equation

$$\begin{aligned}
&(-1)^{g+1}\boldsymbol{t}(z)Q(\zeta \mid \psi_1, \ldots, \psi_{2g+2}) \\
&\qquad = Q(\zeta + \mathrm{i}\gamma|\psi_1, \ldots, \psi_{2g+2}) + Q(\zeta - \mathrm{i}\gamma|\psi_1, \ldots, \psi_{2g+2}).
\end{aligned}$$

In φ-representation the 'components' of these vectors are given by

$$Q(\varphi_1, \ldots, \varphi_{2g+2}|\zeta|\psi_1, \ldots, \psi_{2g+2}) = \mathrm{e}^{\frac{1}{2}(1+\frac{\pi}{\gamma})\zeta + \frac{1}{4\mathrm{i}\gamma}\zeta^2} \prod_{k=1}^{2g+2} \lambda(\zeta|\varphi_k - \psi_k)\langle\varphi_k|\psi_{k-1}\rangle \tag{25}$$

where $\psi_0 \equiv \psi_{2g+2}$,

$$\begin{aligned} \langle\varphi|\psi\rangle &= \mathrm{e}^{\frac{1}{4\mathrm{i}\gamma}(2\varphi\psi-\varphi^2)} \\ \lambda(\zeta|\psi) &= \mathrm{e}^{-\frac{1}{2\mathrm{i}\gamma}\zeta\psi}\Phi(\psi-\zeta)\mathrm{e}^{\frac{\pi+\gamma}{\gamma}(\psi-\zeta)} \end{aligned} \tag{26}$$

and the function $\Phi(\varphi)$ satisfies the functional equation

$$\frac{\Phi(\varphi+\mathrm{i}\gamma)}{\Phi(\varphi-\mathrm{i}\gamma)} = \frac{1}{1+\mathrm{e}^{\varphi}}. \tag{27}$$

The solution to this equation is

$$\Phi(\varphi) = \exp\left(\int_{\mathbb{R}+\mathrm{i}0} \frac{\mathrm{e}^{\mathrm{i}k\varphi}}{4\sinh\gamma k\sinh\pi k}\frac{\mathrm{d}k}{k}\right).$$

This wonderful function and its applications can be found in [9].

As usual we want to consider $Q(\varphi_1,\ldots,\varphi_{2g+2}|\zeta|\psi_1,\ldots,\psi_{2g+2})$ as the kernel of an operator:

$$Q(\varphi_1,\ldots,\varphi_{2g+2}|\zeta|\psi_1,\ldots,\psi_{2g+2}) = \langle\varphi_1,\ldots,\varphi_{2g+2}|\mathcal{Q}(\zeta)|\psi_1,\ldots,\psi_{2g+2}\rangle.$$

The subtle point is that we have to use mixed representations: the vectors $|\psi\rangle$ are the eigenvectors of the operators

$$w \equiv \mathrm{e}^{\psi} = uvu.$$

Notice that this justifies the notation $\langle\varphi|\psi\rangle$ in (26), and that

$$[\psi,\varphi] = 2\mathrm{i}\gamma.$$

The operators $\mathcal{Q}(\zeta)$ satisfy the equations

$$(-1)^{g+1}\boldsymbol{t}(z)\mathcal{Q}(\zeta) = \mathcal{Q}(\zeta+\mathrm{i}\gamma) + \mathcal{Q}(\zeta-\mathrm{i}\gamma). \tag{28}$$

This is the well known Baxter equation.

Before going further, let us discuss the properties of operator $\mathcal{Q}(\zeta)$. We have

$$\begin{aligned} &\overline{\Phi(\varphi)} = \Phi(\bar{\varphi}) \\ &\Phi(\varphi) \sim \exp\left(\frac{1}{4\mathrm{i}\gamma}\varphi^2\right) \qquad \text{as} \quad \varphi\to\infty \end{aligned} \tag{29}$$

so the kernel of $\mathcal{Q}(\zeta)$ for $\zeta\in\mathbb{R}$ is an oscillating function, and it is clear that our operator is well defined on the functions of ψ_j of Schwartz class (Sch_{ψ}) sending them to functions of φ_j which are also of Schwartz class (Sch_{φ}). Using equations (29) one easily finds the kernel $\langle\psi|\mathcal{Q}^*(\zeta)|\varphi\rangle$ of the adjoint operator $\mathcal{Q}^*(\zeta)$ (we consider the case of real ζ). Further, note that the l-operator can be rewritten as

$$l(z) = \frac{1}{\sqrt{z}}\begin{pmatrix} zu & -qu^{-1}w \\ zuw^{-1} & 0 \end{pmatrix}. \tag{30}$$

Applying to this l-operator the same procedure as before one finds that $\mathcal{Q}^*(\zeta)$ also solves the Baxter equation (28):

$$(-1)^{g+1}\boldsymbol{t}(z)\mathcal{Q}^*(\zeta) = \mathcal{Q}^*(\zeta+\mathrm{i}\gamma) + \mathcal{Q}^*(\zeta-\mathrm{i}\gamma).$$

It can be shown that

$$\mathcal{Q}(\zeta) = \mathcal{Q}^*(\zeta) \qquad \text{for} \quad \zeta\in\mathbb{R}.$$

Considering the kernel of operator $\mathcal{Q}^*(\zeta)$ one finds that this operator acts from Sch_{φ} to Sch_{ψ}. So, the products $\mathcal{Q}(\zeta)\mathcal{Q}(\zeta')$ are well defined, at least for $\zeta,\zeta'\in\mathbb{R}$.

We want to show that the operators $\mathcal{Q}(\zeta)$ constitute a commutative family:

$$[\mathcal{Q}(\zeta), \mathcal{Q}(\zeta')] = 0. \tag{31}$$

To this end we want to show that the operator $\mathcal{Q}(\zeta)$ can be rewritten as

$$\mathcal{Q}(\zeta) = \mathrm{tr}_a(\mathcal{L}_{a2q+2}(\zeta)\cdots\mathcal{L}_{a1}(\zeta)) \tag{32}$$

where the operators $\mathcal{L}_{aj}(\zeta)$ act in the tensor product of the 'auxiliary space' labelled by a and of the 'quantum space' where φ_j, ψ_j act. Actually in our case the 'auxiliary space' will be isomorphic to the 'quantum space', i.e. we shall have a universal l-operator. If the operators $\mathcal{L}_{aj}(\zeta)$ satisfy Yang–Baxter equations with some R-matrix then the commutativity (31) follows from the standard argument.

To find the representation (32) rewrite (25) as

$$\mathcal{Q}(\zeta) = \mathrm{e}^{\frac{1}{2}(1+\frac{\pi}{\gamma})\zeta+\frac{1}{4\mathrm{i}\gamma}\zeta^2}\int\prod_{j=1}^{2g+2}\mathrm{d}\varphi'_j\,\mathrm{d}\psi'_j\langle\psi'_j|\mathcal{L}_{aj}(\zeta)|\varphi'_j\rangle\langle\varphi'_j|\psi'_{j-1}\rangle$$

where φ'_j, ψ'_j are operators acting in the 'auxiliary space', $\psi'_0 = \psi'_{2g+2}$. So, (32) indeed takes place if the kernel of the 'universal' l-operator is given by

$$\langle\varphi'|\otimes\langle\psi|\mathcal{L}(\zeta)|\psi'\rangle\otimes|\varphi\rangle = \delta(\varphi-\varphi')\delta(\psi-\psi')\lambda(\zeta|\varphi-\psi).$$

Hence, equation (32) holds for the operators $\mathcal{L}_{aj}(\zeta)$ of the form

$$\mathcal{L}_{12}(\zeta) = \mathcal{P}_{12}\hat{\mathcal{L}}_{12}(\zeta)$$

where $\mathcal{P}_{12}$ is the operator of permutation, and the operator $\hat{\mathcal{L}}_{12}(\zeta)$ acts in the tensor product as follows:

$$\hat{\mathcal{L}}_{12}(\zeta) = \lambda(\zeta \mid \varphi\otimes I - I\otimes\psi).$$

Thus the operator $\mathcal{Q}(\zeta)$ can be considered as trace of the 'universal' monodromy matrix and the commutativity (31) follows from the Yang–Baxter equation:

$$\hat{\mathcal{R}}_{12}(\zeta_1-\zeta_2)\hat{\mathcal{L}}_{23}(\zeta_1)\hat{\mathcal{L}}_{12}(\zeta_2) = \hat{\mathcal{L}}_{23}(\zeta_2)\hat{\mathcal{L}}_{12}(\zeta_1)\hat{\mathcal{R}}_{23}(\zeta_1-\zeta_2) \tag{33}$$

with the simple r-matrix

$$\hat{\mathcal{R}}_{12}(\zeta) = \exp\left(\frac{(I\otimes\psi-\varphi\otimes I)\zeta}{2\mathrm{i}\gamma}\right).$$

The Yang–Baxter equation (33) in our case is almost trivial. In the case of the more general l-operator $\hat{l}(z,\kappa)$ mentioned above, we would need to use a more complicated r-matrix and the proof of Yang–Baxter equations needs some nontrivial identities [10].

The self-adjoint (for real ζ_1, ζ_2) operators $\mathcal{Q}(\zeta_1)$, $\mathcal{Q}(\zeta_2)$ commute, hence the eigenvectors of $\mathcal{Q}(\zeta)$ do not depend on ζ. In fact, the operator $\mathcal{Q}(\zeta)$ is an entire function of ζ. The kernel of $\mathcal{Q}(\zeta)$ has poles, but in the process of analytical continuation the poles never pinch the contour of integration. The Baxter equation (28) implies that $\mathcal{Q}(\zeta_1)$ and $\boldsymbol{t}(z_2)$ also commute. Suppose that $\mathcal{Q}(\zeta)$ ant $t(z)$ are eigenvalues of these operators; owing to equation (28) they satisfy

$$(-1)^{g+1}t(z)\mathcal{Q}(\zeta) = \mathcal{Q}(\zeta+\mathrm{i}\gamma)+\mathcal{Q}(\zeta-\mathrm{i}\gamma). \tag{34}$$

Let us discuss further analytical properties of $\mathcal{Q}(\zeta)$. Since the operator $\mathcal{Q}(\zeta)$ is an entire function of ζ the eigenvalue $\mathcal{Q}(\zeta)$ is an entire function as well. As has been said, $\boldsymbol{t}(0) = \boldsymbol{t}_{g+1}$ belongs to the centre of the algebra defined by the commutation relations (23), so we can fix it. It is convenient to put $\boldsymbol{t}_{g+1} = (-1)^{g+1}2$ which allows us to require that

$$\mathcal{Q}(\zeta)\to 1 \qquad \zeta\to-\infty. \tag{35}$$

From quasiclassical considerations which are completely parallel to those from [3, 8] it is naturally to conjecture that the eigenvalues of $\mathcal{Q}(\zeta)$ have zeros only on the real axis and that asymptotically for $\zeta \to \infty$ one has

$$\mathcal{Q}(\zeta) \sim \mathrm{e}^{-(g+1)(1+\frac{\pi}{\gamma})\zeta} \cos\left(\frac{(g+1)\zeta^2}{\gamma} + \frac{\pi}{4}\right). \tag{36}$$

The important question is whether equation (34) together with the analytical properties described above are sufficient to find the spectrum of commuting Hamiltonians. In our opinion it is impossible: additional information is needed, which is provided in the following section.

7. Duality

Consider the function $\Phi(\zeta)$. The most interesting property of this function is its duality: together with equation (27) it satisfies the equation

$$\frac{\Phi(\varphi + \mathrm{i}\pi)}{\Phi(\varphi - \mathrm{i}\pi)} = \frac{1}{1 + \mathrm{e}^{\frac{\pi}{\gamma}\varphi}}.$$

Using this property and the definition of the operator $\mathcal{Q}(\zeta)$ one finds that there is dual equation for $\mathcal{Q}(\zeta)$:

$$(-1)^{g+1}\boldsymbol{T}(Z)\mathcal{Q}(\zeta) = \mathcal{Q}(\zeta + \pi\mathrm{i}) + \mathcal{Q}(\zeta - \pi\mathrm{i}) \tag{37}$$

where

$$Z = \mathrm{e}^{\frac{2\pi}{\gamma}\zeta}$$

and $\boldsymbol{T}(Z)$ is the trace of the monodromy matrix

$$\tilde{\boldsymbol{M}}(Z) = L_{2g+2}(Z) \cdots L_1(Z)$$

with

$$L(Z) = \frac{1}{\sqrt{Z}} \begin{pmatrix} ZU^{-1} & -QVU \\ ZV^{-1}U^{-1} & 0 \end{pmatrix}.$$

The dual operators

$$U = \mathrm{e}^{\frac{\pi}{\gamma}\varphi} \qquad V = \mathrm{e}^{\pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi}}$$

satisfy the commutation relations

$$UV = QVU$$

with dual

$$Q = \mathrm{e}^{\mathrm{i}\frac{\pi^2}{\gamma}}.$$

The only nontrivial commutation relations of u, v with U, V are

$$uV = -Vu \qquad vU = -Uv$$

which means that

$$S(l(z) \otimes I)(I \otimes L(Z)) = (I \otimes L(Z))(l(z) \otimes I)S$$

with $S = \sigma^3 \otimes \sigma^3$. From here it is obvious that

$$[\boldsymbol{t}(z), \boldsymbol{T}(Z)] = 0.$$

All that is the result of manifest duality of the kernel of $\mathcal{Q}(\zeta)$ with respect to change:

$$\gamma \to \frac{\pi^2}{\gamma} \qquad \zeta \to \frac{\pi}{\gamma}\zeta \qquad \varphi_j \to \frac{\pi}{\gamma}\varphi_j \qquad \psi_j \to \frac{\pi}{\gamma}\psi_j.$$

It is clear that $\boldsymbol{T}(Z_1)$ and $\mathcal{Q}(\zeta_2)$ commute, so equation (37) implies the equation for eigenvalues:

$$(-1)^{g+1}T(Z)\mathcal{Q}(\zeta) = \mathcal{Q}(\zeta + \pi \mathrm{i}) + \mathcal{Q}(\zeta - \pi \mathrm{i}). \tag{38}$$

The function $\mathcal{Q}(\zeta)$ is not an entire function of z as is the case in other situations (for example, [11]), which is why equation (28) alone does not look strong enough to define it. However, equation (37) controlling the behaviour of $\mathcal{Q}(\zeta)$ under 2πi-rotation in the z-plane must provide the missing information.

So, our main conjecture is as follows.

Conjecture 4. *The spectrum on $\boldsymbol{t}(z)$ (and, simultaneously, of $\boldsymbol{T}(Z)$) is described by all solutions of equations (34) and (38) such that*

(1) $t(z)$ and $T(Z)$ are polynomials of degree $g+1$.
(2) $\mathcal{Q}(\zeta)$ is an entire function of ζ.
(3) $\mathcal{Q}(\zeta)$ satisfies (35) and (36).
(4) All the zeros of $\mathcal{Q}(\zeta)$ in the strip $-(\pi+\gamma) < \mathrm{Im}\,(\zeta) < (\pi+\gamma)$ are real.

8. Separation of variables

The relation of integrable models to the algebraic geometry can be completely understood in the framework of separation of variables.

We have already mentioned that

$$[\boldsymbol{b}(z), \boldsymbol{b}(z')] = 0$$

which implies commutativity of the operators $\boldsymbol{z}_j$ defined as roots of $\boldsymbol{b}(z)$. Consider the operators

$$\boldsymbol{w}_j = (-1)^{g+1} q\, \boldsymbol{d}(\overleftarrow{\boldsymbol{z}}_j)$$

where $d(\overleftarrow{\boldsymbol{z}_j})$ means that $\boldsymbol{z}_j$, which does not commute with coefficients of $\boldsymbol{d}(z)$, is substituted to this polynomial from the left. Following Sklyanin [4] one shows that

$$\boldsymbol{z}_j \boldsymbol{w}_k = \boldsymbol{w}_k \boldsymbol{z}_j \qquad j \neq k \qquad \boldsymbol{z}_j \boldsymbol{w}_j = q^2 \boldsymbol{w}_j \boldsymbol{z}_j$$

and

$$\boldsymbol{w}_j^2 - \boldsymbol{w}_j \boldsymbol{t}(\overleftarrow{\boldsymbol{z}}_j) + 1 = 0. \tag{39}$$

Introduce the operators

$$\boldsymbol{\zeta}_j = \tfrac{1}{2}\log(\boldsymbol{z}_j)$$

and consider the wavefunction corresponding to a given set of eigenvalues of integral of motion $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_g$ in $\boldsymbol{\zeta}$-representation. Equation (39) implies [4] that we can look for this wavefunction in the form

$$\langle \zeta_1, \ldots, \zeta_g | t_1, \ldots, t_g \rangle = \mathcal{Q}(\zeta_1) \cdots \mathcal{Q}(\zeta_g)$$

where $\mathcal{Q}(\zeta)$ satisfies

$$\mathcal{Q}(\zeta + \mathrm{i}\gamma) + \mathcal{Q}(\zeta - \mathrm{i}\gamma) = (-1)^{g+1} t(z) \mathcal{Q}(\zeta)$$

where $t(z)$ is constructed from the eigenvalues t. This equation coincides with equation (28) written for particular eigenvalues. So, following [8] we claim that the wavefunction in separated variables is defined by the eigenvalue of the operator $\mathcal{Q}$ which connects two different approaches to integrable models.

Note that the vector $|t_1, \ldots, t_g\rangle$ is the eigenvector for the operators $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_g$ since the function $\mathcal{Q}(\zeta)$ satisfies equation (37). In order to identify explicitly the eigenvalues of $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_g$ we shall write $|t_1, \ldots, t_g; T_1, \ldots, T_g\rangle$.

We have the algebra of operators $\mathcal{A}(q)$ and the dual algebra $\mathcal{A}(Q)$ which act in the same space $\mathfrak{H}$. All the operators from $\mathcal{A}(q)$ commute with the operators from $\mathcal{A}(Q)$. The fundamental property of $\mathcal{A}(q)$ is that it is spanned as linear space by elements of the form (17) according to conjecture 3. A similar fact must be true for $\mathcal{A}(Q)$. Taking these facts together one realizes the algebra $\mathcal{A}(q) \cdot \mathcal{A}(Q)$ is spanned by the elements of the form

$$\begin{aligned} \mathcal{X} = xX = p_L(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_g) P_L(\boldsymbol{T}_1, \ldots, \boldsymbol{T}_g) \\ \times g(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_g) G(\boldsymbol{B}_1, \ldots, \boldsymbol{B}_g) P_R(\boldsymbol{T}_1, \ldots, \boldsymbol{T}_g) p_R(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_g). \end{aligned} \tag{40}$$

We denote by $h(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_g)$ and $H(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_g)$ the antisymmetric polynomials obtained from $g(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_g)$ and $G(\boldsymbol{B}_1, \ldots, \boldsymbol{B}_g)$: for example,

$$h(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_g) = \prod \boldsymbol{z}_i \prod_{i<j} (\boldsymbol{z}_i - \boldsymbol{z}_j) g(\boldsymbol{b}_1(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_g), \ldots, \boldsymbol{b}_g(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_g)).$$

Let us consider the matrix element of operator $\mathcal{X}$ between two eigenvectors of Hamiltonians. The wavefunctions are real for real ζ. By the requirement of self-adjointness of $\boldsymbol{t}(z)$ and $\boldsymbol{T}(Z)$ one defines the scalar product [4]. The matrix element in question is

$$\begin{aligned} \langle t_1, \ldots, t_g; T_1, \ldots, T_g | \mathcal{X} | t'_1, \ldots, t'_g; T'_1, \ldots, T'_g \rangle \\ = p_L(t_1, \ldots, t_g) p_R(t'_1, \ldots, t'_g) P_L(T_1, \ldots, T_g) P_R(T'_1, \ldots, T'_g) \\ \times \int_{-\infty}^{\infty} \mathrm{d}\zeta_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}\zeta_g h(z_1, \ldots, z_g) H(Z_1, \ldots, Z_g) \prod_{j=1}^{g} \mathcal{Q}(\zeta_j) \mathcal{Q}'(\zeta_j). \end{aligned} \tag{41}$$

When does the integral for the matrix element (41) converge? Suppose that

$$h \sim z_j^{g+k+1} \qquad H \sim Z_j^{g+l+1} \qquad \text{when} \quad \zeta_j \to \infty.$$

Then the integrand in the matrix element behaves when $\zeta_j \to \infty$ as

$$\exp 2\zeta_j \left((k-1) + \frac{\pi}{\gamma}(l-1) \right).$$

Hence, for generic γ the integral converges only if $k = 1, l = 0$ or $k = 0, l = 1$. When γ is small we can allow the operators with $l = 0$ and $k < \frac{\pi}{\gamma}$; oppositely, when γ is big the operators with $k = 0$ and $l < \frac{\gamma}{\pi}$ are allowed. The limits $\gamma \to 0$ and $\gamma \to \infty$ are two dual quasiclassical limits. For these limits the operators $l = 0, \forall k$ and $k = 0, \forall l$ respectively define the classical observables. At least these operators must be defined in the quantum case: if the quantization procedure makes sense the principle of correspondence must hold. Hence, the fact that in general only two operators with $k = 1, l = 0$ or $k = 0, l = 1$ lead to convergent integrals means that some regularization of these integrals is needed. The regularized integrals in question must allow us to define the matrix element (41) for arbitrary k, l; they have to coincide with usual integrals whenever the latter are applicable and they must satisfy some additional requirements which will be discussed in the section 9. The origin of these additional requirements is in the cohomological construction explained in section 4.

Note that any antisymmetric with respect to $z_1, \ldots, z_g$ and $Z_1, \ldots, Z_g$ polynomial

$$h(z_1, \ldots, z_g) H(Z_1, \ldots, Z_g)$$

can be presented as a linear combination of products of Schur-type determinants

$$\det |z_i^{k_j}| \det |Z_i^{l_j}|$$

where $\{k_1, \dots, k_g\}$ and $\{l_1, \dots, l_g\}$ are arbitrary sets of positive integers. So, the integrals (41) can be expressed in terms of 1-fold integrals

$$\langle l|L\rangle \simeq \int_{-\infty}^{\infty} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)l(z)L(Z)\,\mathrm{d}\zeta \tag{42}$$

where l and L are polynomials such that $l(0) = 0$ and $L(0) = 0$. The symbol $\simeq$ means that the integrals in the rhs are not always defined; the regularization is defined in appendix B. In the next section we describe results of this regularization.

9. Deformed Abelian differentials

In appendix B we define the polynomials $s_k(z)$. These polynomials are of the form

$$\begin{aligned} s_k(z) &= z^{g+1+k} \qquad -g \leqslant k \leqslant 0 \\ s_k(z) &= \frac{1}{\mathrm{i}\gamma}\left(\frac{q^k-1}{q^k+1}\right) z^{g+1+k} + \cdots \qquad k \geqslant 1 \end{aligned} \tag{43}$$

where $\cdots$ stands for terms of lower degree (containing t_j, t'_j in coefficients) explicitly given in appendix B.

In the classical case every polynomial defines an Abelian differential on the affine curve $X - \infty^{\pm}$. Similarly, we consider the polynomials s_k as corresponding to 'deformed Abelian differentials'. Let us be more precise. The regularized integrals are defined in appendix B in such a way that they satisfy several conditions. The first of them is

$$\langle s_k|S_l\rangle = 0 \qquad k \geqslant g+1 \qquad \forall l \tag{44}$$

$$\langle s_k|S_l\rangle = 0 \qquad \forall k \qquad l \geqslant g+1. \tag{45}$$

Owing to (44), we consider the polynomials s_k, $k \geqslant g+1$ as corresponding to exact forms. The polynomials s_k with $k = -1, \dots, -g$ correspond to first-kind differentials, s_o corresponds to the third-kind one and s_k with $k = 1, \dots, g$ correspond to second-kind differentials.

Explicitly, the relation with the classical case is as follows. Consider the case $t(z) = t'(z)$ and take the limit

$$r_k = z^{-1} \lim_{\gamma\to 0} s_k(z).$$

Then the classical Abelian differential related to s_k is

$$\mu_k = \frac{r_k(z)}{y}\,\mathrm{d}z.$$

A similar interpretation can be given to S_k which correspond to Abelian differentials in the dual classical limit $\gamma \to \infty$. However, the most interesting feature of our construction is that together with this cohomological interpretation an alternative 'homological' one is possible. The polynomials S_k, $k \geqslant g+1$ correspond to retractable cycles according to (45). The polynomials S_k for $k = \pm 1, \dots, \pm g$ are interpreted as analogues of the cycles δ_k on the 'deformed affine curve'; S_0 corresponds to cycle δ_0 around ∞^+ which is nontrivial on the affine curve. The pairing $\langle l|L\rangle$ defines the integral of the differential defined by l over the cycle defined by L. The asymptotics of the integrals $\langle l|L\rangle$ in the classical limit $\gamma \to 0$ are, indeed, described by Abelian integrals. Certainly, the opposite interpretation (l-cycle, L-differential) is possible, which corresponds to the dual classical limit. It is not the first time that this kind of object has appeared [5], but it is the first time that we have observed real duality between two classical limits.

Let us define the pairing between two polynomials l_1 and l_2:

$$l_1 \circ l_2 = \lim_{\Lambda\to\infty} \int_{\Lambda}^{\Lambda+\mathrm{i}\pi} [\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta) l_1(z)\delta_\gamma^{-1}(\mathcal{Q}\mathcal{Q}'l_2)(\zeta - \mathrm{i}\pi) + \mathcal{Q}(\zeta - \pi\mathrm{i})\mathcal{Q}'(\zeta - \mathrm{i}\pi) l_1(z)\delta_\gamma^{-1}(\mathcal{Q}\mathcal{Q}'l_2)(\zeta - \mathrm{i}\gamma)]\,\mathrm{d}\zeta \tag{46}$$

where

$$\delta_\xi(f(\zeta)) = f(\zeta + \mathrm{i}\xi) - f(\zeta). \tag{47}$$

One can show that these formulae give well-defined antisymmetric pairings which correspond classically to natural pairing between meromorphic differentials

$$\omega_1 \circ \omega_2 = \mathrm{res}_{p=\infty^+}\left(\omega_1(p)\int^p \omega_2\right).$$

The polynomials $s_{\pm j}$ and for $j = 1, \ldots, g$ constitute the canonical basis

$$s_k \circ s_l = \operatorname{sgn}(k - l)\delta_{k,-l}.$$

Similarly, to introduce the definition of $L_1 \circ L_2$ it is sufficient to make the necessary replacements in (46): $l_i \leftrightarrow L_i$, $z \leftrightarrow Z$, $\gamma \leftrightarrow \pi$. The polynomials $S_{\pm j}$ are canonically conjugated.

The following antisymmetric polynomials play the role of 2-forms σ and σ' used in classics:

$$\begin{aligned} c(z_1, z_2) &= \sum_{j=1}^{g}(s_{-j}(z_1)s_j(z_2) - s_j(z_1)s_{-j}(z_2)) \\ C(Z_1, Z_2) &= \sum_{j=1}^{g}(S_{-j}(Z_1)S_j(Z_2) - S_j(Z_1)S_{-j}(Z_2)). \end{aligned} \tag{48}$$

As usual [5], the most important property of the deformed Abelian integrals is that the Riemann bilinear relations remain valid after the deformation. Namely, consider the following $2g \times 2g$ period matrix: P with the matrix elements

$$P_{kl} = \langle s_k | S_l \rangle \qquad k, l = -g, \ldots, -1, 1, \ldots, g.$$

The deformed Riemann bilinear identity is formulated as follows.

Proposition 2. *The matrix P belongs to the symplectic group:*

$$P \in Sp(2g). \tag{49}$$

This proposition is equivalent to a number of bilinear relations between the deformed Abelian integrals. To prove them it is convenient to consider the domain of small γ ($\gamma < \pi/n$) when the regularization of integrals simplifies, and then continue analytically with respect to γ. Nevertheless, the proof is rather complicated technically: it is based on nontrivial properties of the regularized integrals. We do not give this bulky proof here.

There is one more relation for deformed Abelian integrals. One can check that

$$\langle d | S_k \rangle = 0 \qquad \forall k \qquad d = \sum_{j=1}^{g}(t_j - t'_j)s_{-j} \tag{50}$$

$$\langle s_k | D \rangle = 0 \qquad \forall k \qquad D = \sum_{j=1}^{g}(T_j - T'_j)S_{-j}. \tag{51}$$

Relations (50) do not have a direct analogue in terms of Abelian integrals; recall that we put $t_j = t'_j$ taking the classical limit which turns the relation into triviality. However, there is another way of taking the classical limit where this equation is important [3].

10. Return to quantization of the affine Jacobian

Let us return to the main subject of this paper: quantization of the affine Jacobian. Consider any $x \in \mathcal{A}(q)$. Such x is identified with $x \cdot I \in \mathcal{A}(q) \cdot \mathcal{A}(Q)$, so, owing to conjecture 3 the matrix element of x can be presented as

$$\langle t_1, \ldots, t_g | x | t'_1, \ldots, t'_g \rangle = p_L(t_1, \ldots, t_g) p_R(t'_1, \ldots, t'_g) \\ \times \int_{-\infty}^{\infty} \mathrm{d}\zeta_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}\zeta_g h(z_1, \ldots, z_g) H_I(Z_1, \ldots, Z_g) \prod_{j=1}^{g} \mathcal{Q}(\zeta_j) \mathcal{Q}'(\zeta_j) \tag{52}$$

where the eigenvalues of $\boldsymbol{T}_j$ are the same on the left and the right, so we do not write them explicitly; the polynomial H_I which corresponds to $X = I$ is given by

$$H_I(Z_1, \ldots, Z_g) = \prod_{j=1}^{g} Z_j \prod_{i<j} (Z_i - Z_j).$$

Note that

$$H_I = S_{-1} \wedge \cdots \wedge S_{-g}. \tag{53}$$

The formula for the matrix elements (52) for small γ (when no regularization of integrals is needed) can be deduced rigorously starting from the realization of $\mathcal{A}(q)$ defined in section 4. The following equations follow respectively from (44), (50), (49) (recall the notation of section 4):

$$s_k \wedge \mathcal{V}^{g-1} \simeq 0 \qquad k \geqslant g + 1 \tag{54}$$

$$c \wedge \mathcal{V}^{g-2} \simeq 0 \tag{55}$$

$$d \wedge \mathcal{V}^{g-1} \simeq 0 \tag{56}$$

where $\simeq$ means that these expressions vanish, being substitute into the integral (52). Equation (55) needs explanation. To prove this equation one has to take into account the Riemann bilinear identity (49) and equation (53); note that

$$S_{-i} \circ S_{-j} = 0 \qquad 1 \leqslant i \qquad j \leqslant g.$$

The formula for the matrix elements (52) can be rigorously deduced for small γ. Hence, equations (54)–(56) lead to certain equations for the operators from $\mathcal{A}(q) \cdot \mathcal{A}(Q)$. The latter equations are obtained by applying the operation χ (section 4):

$$\chi(s_k \wedge \mathcal{V}^{g-1}) = 0 \qquad k \geqslant g + 1 \tag{57}$$

$$\chi(c \wedge \mathcal{V}^{g-2}) = 0 \tag{58}$$

$$\chi(d \wedge \mathcal{V}^{k-1}) = 0. \tag{59}$$

We conclude that the formulae for the polynomials s_k needed in section 4 are exactly the same as given in (43). Thus we put together the algebraic part of this work with the analytical one.

On the other hand, equations (57)–(59) are of purely algebraic character, so if they are valid for small γ they must be valid always. That is why we regularized the integrals for the matrix element in order that equations (54)–(56) hold for any γ.

Moreover, there is a dual model and we can consider the operators $\mathcal{X} = xX$ from $\mathcal{A}(q) \cdot \mathcal{A}(Q)$. Equations (57)–(59) and the dual equations still have to be valid. The regularized integrals are defined in such a way that this is the case. Equations (57), (59) and their duals clearly follow from (44), (45) and (50), (51). The most interesting is equation (58). Owing to the Riemann bilinear relation this equation follows from

$$c \wedge \mathcal{V}^{g-2} \simeq 0 \tag{60}$$

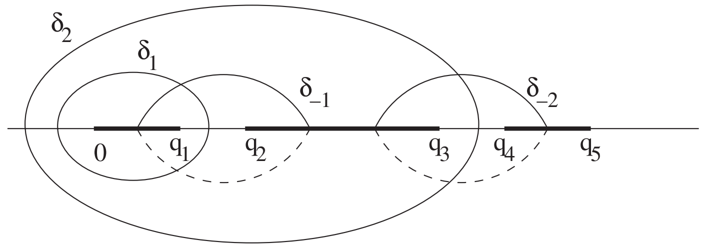

Figure 1.

which is true if the subspace $c \wedge \mathcal{V}^{g-2}$ of the space $\mathcal{V}^g$ is convoluted with the subspace

$$\frac{\mathcal{V}_g}{C \wedge \mathcal{V}_{g-2}}$$

where $\mathcal{V}_k$ is the same as $\mathcal{V}^k$ for the dual model (this notation is not occasional: the space $\mathcal{V}_k$ plays the role of k-cycles for k-forms from $\mathcal{V}^k$). In other words, we impose the equation

$$C \wedge \mathcal{V}_{g-2} = 0$$

and the dual equation (60) is imposed automatically due to the Riemann bilinear relation.

Let us discuss the classical limit in some more detail. Consider the hyper-elliptic curve X. If we realize this curve as a characteristic equation of the classical analogue of the monodromy matrix $\tilde{\boldsymbol{m}}(z)$ (21), the branch points of the curve can be shown real non-negative. In fact, requiring $t_{g+1} = (-1)^{g+1}2$ we put one of the branch points at $z = 0$. Thus the branch points are $0 = q_1 < \cdots < q_{2g+2}$. The Riemann surface is realized as a two-sheet covering of the plane of z with cuts $I_k = [q_{2k-1}, q_{2k}]$, $k = 1, \ldots, g + 1$. The canonical a-cycles δ_{-j} and b-cycles δ_j are shown in figure 1.

Under classical dynamics each of the separate variables z_j oscillates in the interval $q_{2j-1} \leqslant z_j \leqslant q_{2j}$; topologically this corresponds to motion along the a-cycle δ_{-j}. One can show that the integral $\langle s_k \, | S_{-j} \rangle$ is described in the classical limit $\gamma \to 0$ by δ_{-j} of differential μ_k. Thus the g-cycle (53) corresponds to the classical trajectory $\delta_{-1} \wedge \cdots \wedge \delta_{-g}$. Recall that the cycle (53) corresponds to insertion of the unit operator of the dual model. Introducing other dual operators, one gets integrals with respect to both a-cycles and b-cycles. Classically, the corresponding trajectories are not real, but the factorization by $\sigma' \wedge W_{m-2}$ in (12) guarantees that the classical non-real trajectories are not singular. The topological interpretation of the dual model is a good point to conclude this paper.

Appendix A

In this appendix we shall give the canonical definition of the affine Jacobi variety $J_{\mathrm{aff}}(t)$. Consider the hyper-elliptic curve X of genus g:

$$w^2 - t(z)w + 1 = 0.$$

We have the canonical basis with a-cycles δ_k, $-g \leqslant k < 0$ and b-cycles δ_k, $0 < k \leqslant g$. Associate with this basis the basis of normalized holomorphic differentials ω_j:

$$\int_{\delta_{-i}} \omega_j = \delta_{ij} \qquad B_{ij} = \int_{\delta_i} \omega_j .$$

The Jacobi variety of this curve is the g-dimensional complex torus:

$$J(t) = \frac{\mathbb{C}^g}{\mathbb{Z}^g \times B\mathbb{Z}^g}.$$

With every point $p \in X$ we identify the point $\alpha(p) \in J(t)$ with coordinates

$$\alpha_j(p) = \int_b^p \omega_j.$$

For the reference point b it is convenient to take one of the branch points. The curve X has two points over the point $z = \infty$; denote them by $\infty^\pm$ and consider the $(g-1)$-dimensional subvariety of $J(t)$ defined by

$$\Theta^\pm = \{\zeta \in J(t) | \theta(\zeta + \alpha(\infty^-))\theta(\zeta + \alpha(\infty^+)) = 0\}$$

where θ is Riemann theta-function. It can be shown that there exist an isomorphism:

$$J_{\mathrm{aff}}(t) \simeq J(t) - \Theta^\pm. \tag{61}$$

The equivalence of this description with the description in terms of divisors (section 1) is due to the Abel map $X[g] \to J(t)$ explicitly given by

$$\zeta = \alpha(\mathcal{P}) + \Delta \qquad \alpha(\mathcal{P}) = \sum \alpha(p_j)$$

where Δ is the Riemann characteristic.

Appendix B

In this appendix we describe the regularization of integrals which has been used in the paper.

Define

$$\Delta_\xi(f(\zeta)) = f(\zeta + \mathrm{i}\xi) - f(\zeta - \mathrm{i}\xi).$$

Introduce the polynomials

$$\begin{aligned}
s_k(z) = \frac{1}{2\mathrm{i}\gamma}\{ & t(z)\Delta_\gamma^{-1}[z^{k-g-1}t(z)]_> + t'(z)\Delta_\gamma^{-1}[z^{k-g-1}t'(z)]_> \\
& - t(z)\Delta_\gamma^{-1}[z^{k-g-1}q^{2(g+1-k)}t'(zq^{-2})]_> \\
& - t'(z)\Delta_\gamma^{-1}[z^{k-g-1}q^{2(g+1-k)}t(zq^{-2})]_> \\
& - \tfrac{1}{2}(t'(z)[z^{k-g-1}t(z)]_> + t(z)[z^{k-g-1}t'(z)]_>) \\
& + (q^{2(g+1-k)k} - q^{2(k-g-1)})[z^{k-g-1}]_>\} \qquad k \geqslant 0 \\
s_k(z) = z^{g+1+k} & \qquad -g \leqslant k \leqslant 0
\end{aligned}$$

where the notation $[\ ]_>$ means that only the positive degrees of Laurent series in brackets are taken. Obviously, $\deg(s_k) = g + 1 + k$. Further, with every function $f(\zeta)$ associate the functions:

$$\begin{aligned}
u[f](\zeta) = \frac{1}{2\mathrm{i}\gamma}\{ & t(z)\Delta_\gamma^{-1}(f(\zeta)t(z)) + t'(z)\Delta_\gamma^{-1}(f(\zeta)t'(z)) \\
& - t(z)\Delta_\gamma^{-1}(f(\zeta - \mathrm{i}\gamma)t'(zq^{-2})) - t'(z)\Delta_\gamma^{-1}(f(\zeta - \mathrm{i}\gamma)t(zq^{-2})) \\
& - f(\zeta)t(z)t'(z) + f(\zeta + \mathrm{i}\gamma) - f(\zeta - \mathrm{i}\gamma)\} \\
v[f](\zeta) = \frac{1}{2\mathrm{i}\gamma}\{ & (-1)^{g+1}(\Delta_\gamma^{-1}(f(\zeta - \mathrm{i}\gamma)t(zq^{-2}))\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta - \mathrm{i}\gamma) \\
& + \Delta_\gamma^{-1}(f(\zeta - \mathrm{i}\gamma)t'(zq^{-2}))\mathcal{Q}(\zeta - \mathrm{i}\gamma)\mathcal{Q}'(\zeta) \\
& - \Delta_\gamma^{-1}(f(\zeta)t(z))\mathcal{Q}(\zeta - \mathrm{i}\gamma)\mathcal{Q}'(\zeta) \\
& - \Delta_\gamma^{-1}(f(\zeta)t'(z))\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta - \mathrm{i}\gamma)) \\
& + f(\zeta)\mathcal{Q}(\zeta - \mathrm{i}\gamma)\mathcal{Q}'(\zeta - \mathrm{i}\gamma) + f(\zeta - \mathrm{i}\gamma)\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)\}.
\end{aligned}$$

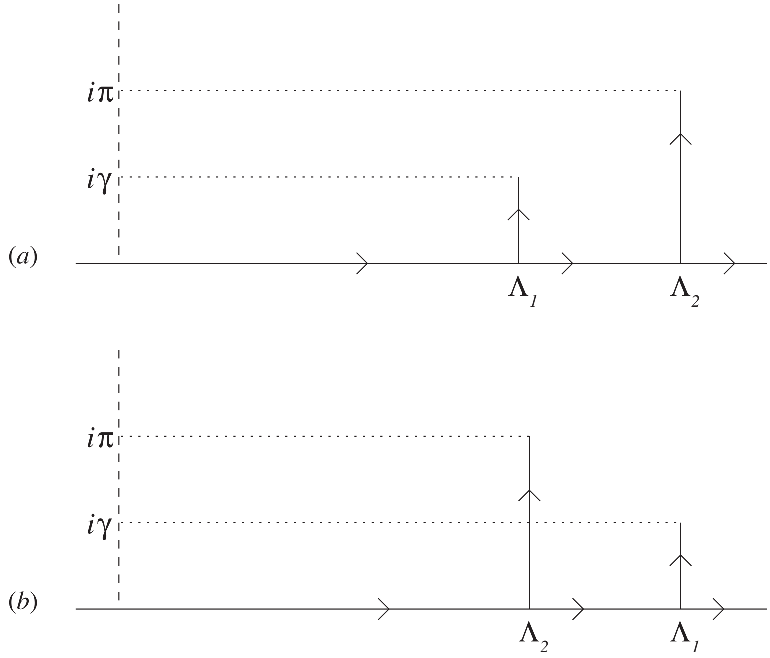

Figure B1.

Define

$$s_k^-(\zeta) = \begin{cases} -s_k(z) + u[f](\zeta) & f = z^{k-g-1} \qquad k \geqslant 1 \\ -s_0(z) + u[f](\zeta) + (-1)^{g+1}2 & f = \zeta z^{-g-1} \qquad k = 0 \\ -z^{g+1+k} & -g \leqslant k \leqslant -1 \end{cases}$$

and

$$p_k(\zeta) = \begin{cases} v[f](\zeta) & f = z^{k-g-1} \qquad k \geqslant 1 \\ v[f](\zeta) & f = \zeta z^{-g-1} \qquad k = 0 \\ 0 & -g \leqslant k \leqslant -1. \end{cases}$$

These definitions imply that

$$(s_k(z) + s_k^-(z))\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta) = \delta_\gamma(p_k(\zeta)). \tag{62}$$

Similarly, one introduces the functions $S_k(Z)$, $S_k^-(\zeta)$, $P_k(\zeta)$, changing everywhere z by Z, q by Q and iγ-shift of ζ by iπ-shift of ζ. One has

$$(S_k(z) + S_k^-(z))\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta) = \delta_\pi(P_k(\zeta)). \tag{63}$$

Our goal is to define a pairing $\langle l|L\rangle$ between two arbitrary polynomials $l(z)$ and $L(Z)$ such that $l(0) = 0$, $L(0) = 0$. Note that every such polynomial l (L) can be presented as a linear combination of polynomials s_k (S_k).

Consider figure B1.

We define

$$\langle s_k|S_l\rangle \equiv \int_{-\infty}^{\Lambda_1} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k(z)S_l(Z)\,\mathrm{d}\zeta$$
$$+ \int_{\Lambda_1}^{\Lambda_2} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k^-(\zeta)S_l(Z)\,\mathrm{d}\zeta + \int_{\Lambda_2}^{\infty} \mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k^-(\zeta)S_l^-(\zeta)\,\mathrm{d}\zeta$$

$$-\int_{\Lambda_1}^{\Lambda_1+\mathrm{i}\gamma} S_l(Z)p_k(\zeta)\,\mathrm{d}\zeta - \int_{\Lambda_2}^{\Lambda_2+\mathrm{i}\pi} s_k^-(\zeta)P_l(\zeta)\,\mathrm{d}\zeta \tag{64}$$

see figure B1(a). The first integral in the rhs converges at $-\infty$ because $l(0) = L(0) = 0$. Equations (62), (63) guarantee that the regularization (64) does not depend on Λ_1, Λ_2 if they remain ordered: $\Lambda_1 < \Lambda_2$. Moreover, let us transform figure B1(a) into figure B1(b). The alternative definition of the regularized integral referring to figure B1(b) is

$$\begin{aligned}\langle s_k|S_l\rangle \equiv \int_{-\infty}^{\Lambda_2} &\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k(z)S_l(Z)\,\mathrm{d}\zeta \\ &+\int_{\Lambda_2}^{\Lambda_1}\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k(z)S_l^-(\zeta)\,\mathrm{d}\zeta + \int_{\Lambda_1}^{\infty}\mathcal{Q}(\zeta)\mathcal{Q}'(\zeta)s_k^-(\zeta)S_l^-(\zeta)\,\mathrm{d}\zeta \\ &-\int_{\Lambda_1}^{\Lambda_1+\mathrm{i}\gamma} s_k(z)P_l(\zeta)\,\mathrm{d}\zeta - \int_{\Lambda_2}^{\Lambda_2+\mathrm{i}\pi} S_l^-(\zeta)p_k(\zeta)\,\mathrm{d}\zeta.\end{aligned} \tag{65}$$

The equivalence of the regularizations (64) and (65) is based on the following fact. It is easy to realize that for any l and L there exists a function $X_{kl}(\zeta)$ such that

$$(S_l(Z) - S_l^-(\zeta))p_k(\zeta) = \delta_\pi(X_{kl}(\zeta))$$
$$(s_k(z) - s_k^-(\zeta))P_l(\zeta) = \delta_\gamma(X_{kl}(\zeta)).$$

The equivalence in question follows from the equality:

$$\begin{aligned}\int_{\Lambda}^{\Lambda+\mathrm{i}\gamma}(S_l(Z) - S_l^-(\zeta))g(\zeta)\,\mathrm{d}\zeta &= \left(\int_{\Lambda+\mathrm{i}\pi}^{\Lambda+\mathrm{i}\pi+\mathrm{i}\gamma} - \int_{\Lambda}^{\Lambda+\mathrm{i}\gamma}\right)X_{kl}(\zeta)\,\mathrm{d}\zeta \\ &= \left(\int_{\Lambda+\mathrm{i}\gamma}^{\Lambda+\mathrm{i}\pi+\mathrm{i}\gamma} - \int_{\Lambda}^{\Lambda+\mathrm{i}\pi}\right)X_{kl}(\zeta)\,\mathrm{d}\zeta = \int_{\Lambda}^{\Lambda+\mathrm{i}\pi}(s_k(z) - s_k^-(\zeta))G(\zeta)\,\mathrm{d}\zeta.\end{aligned}$$

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